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Research article

Fundamental Properties of Digital Simplicial Homology Groups

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Abstract

In this article we give characteristic properties of the simplicial homology groups of digital images which are based on the simplicial homology groups of topological spaces in algebraic topology. We then investigate Eilenberg-Steenrod axioms for the simplicial homology groups of digital images. We state universal coefficient theorem for digital images. We conclude that the Künneth formula for simplicial homology doesn't be hold in digital images. Moreover we show that the Hurewicz Theorem need not be hold in digital images.

Keywords: Digital image, Digital simplicial homology group, Künneth formula, Hurewicz theorem.

1. Introduction

Homology theory is a branch of algebraic topology that attempts to distinguish between spaces by constructing algebraic invariants that reflect the properties of a space. The homology groups of a space characterize the number and type of holes in that space and therefore give a fundamental description of its structure. This type of information is used, for example, for determining similarities between proteins in molecular biology (see [10]). As a result, the study of computational homology is an active area of researches.

There are several [21] different ways of defining homology groups. The simplicial homology theory is a classical one. Simplicial homology theory [25] is defined by means of a covariant functor from the category of simplicial complexes to the category of chain complexes. The advantages of this theory are that some proofs appear too easy and are that it is usually easy to compute $H_q(X)$ for specific X space. The simplicial theory [9] is easy to calculate with, whereas in the singular theory it is quite hard to calculate the homology of many spaces. On the other hand, many of the theorems that have been proved for the singular theory such as the existence of induced homomorphisms, their homotopy invariance, and the connection between homology and homotopy groups can also be proved for the simplicial theory. Simplicial homology theory applies to the category of pairs (X, A) where X and A have triangulations K and L for which L is a subcomplex of K, while the singular homology theory applies to all pairs (X, A) where X is a topological space with subspace A.

Simplicial homology is the easiest homology theory to adapt for implementation on a computer. This homology is feasible for application to real life situations, such as image analysis, medical imaging, and data analysis in general. Since computer graphics, digital imaging and visualization are depended on the describing of triangulated surfaces, simplices are significant building blocks.

Eilenberg and Steenrod [11] obtained a system of axioms which characterize homology theories with simplices and singular chains. Eilenberg and Steenrod showed that both simplicial and singular homology have these axioms. They also proved that, on triangulable spaces, every homology theory having these properties would give the same answer.

Kaczynski et al. [16] proposed to compute homology groups with a sequence of reductions. The idea is to derive a new object with less cells while preserving homology at each step of the transformation. During the computations, to ensure invertible coefficients, Kaczynski et al. choose them in a field.

Kaczynski, Mischaikow and Mrozek [17] show that homology is computable by presenting in detail an algorithm based on linear algebra over the integers. This is essential because it demonstrates that the homology group of a topological space made up of cubes is computable. However, for spaces made up of many cubes, the algorithm is of little immediate practical value. Therefore, they introduce combinatorial techniques for reducing the number of elements involved in the computation.

Peltier, Alayrangues, Fuchs and Lachaud [22] focus on homology groups, which are known to be computable in finite dimensions, and which have a good topological characterization power at least in low dimensions. For instance Euler characteristic and Betti numbers are straightforwardly deduced from homology groups. They choose simplicial homology since it is widely used in geometric modeling and is straightforwardly applicable to digital objects. After that, they present approach for computing homology groups. They also show some experiments, both on simplicial and discrete objects.

Arslan et al. [1] introduce the simplicial homology groups of n-dimensional digital images from algebraic topology. They also compute simplicial homology groups of MSS_{18}. Boxer, Karaca and Oztel [8] expand knowledge of the simplicial homology groups of digital images. They study the simplicial homology groups of certain minimal simple closed surfaces, extend an earlier definition of the Euler characteristics of digital image, and compute the Euler characteristic of several digital surfaces.

Karaca and Ege [18] present the digital cubical homology groups of digital images. They study some fundamental properties of digital images. They show that the Mayer-Vietoris Theorem does not hold in digital images.

Karaca and Ege [19] study some results related to the simplicial homology groups of 2D digital images. They show that if a bounded digital image is nonempty and -connected, then its homology groups at the first dimension are a trivial group. They also prove that the homology groups of the operands of a wedge of digital images need not be additive.

This paper is organized as follows: In Section 2 we introduce the general notions of digital images with κ-adjacency relations, we give definitions and theorems that are related to digital homotopy, digital fundamental group. In Section 3 we present some fundamental properties and definitions with respect to digital simplicial homology groups. In Section 4 we give the Eilenberg-Steenrod axioms for digital images and we conclude that excision axiom doesn't exist in digital images. In Section 5 we express Universal Coefficient Theorem for digital images and we give an example about this theorem. In Section 6 we show that the Künneth formula for simplicial homology doesn't be hold in digital images. In Section 7 we show that the Hurewicz Theorem need not be hold in digital images.

2. Preliminaries

Let $\mathbb{Z}$ be the set of integers. *A (binary) digital image* is a pair (X,κ), where $X \subset \mathbb{Z}^n$ for some positive integer n and κ represents certain adjacency relation for the members of X. Various adjacency relations are used in the study of digital images. Some of the well-known adjacencies are the following. Two points p and q in $\mathbb{Z}^2$ are *8-adjacent* if they are distinct and differ by at most 1 in each coordinate; points p and q in $\mathbb{Z}^2$ are *4-adjacent* if they are 8-adjacent and differ in exactly one coordinate. Two points p and q in $\mathbb{Z}^3$ are *26-adjacent* if they are distinct and differ by at most 1 in each coordinate; they are *18-adjacent* if they are 26-adjacent and differ in at most two coordinate; they are *6-adjacent* if they are 18-adjacent and differ in exactly one coordinate. We generalize these adjacencies as follows. Let l, n be positive integers, $1 \le l \le n$ and two distinct points $p=(p_1,p_2,\ldots,p_n), q=(q_1,q_2,\ldots,q_n)$ in $\mathbb{Z}^n$, p and q are k_l*-adjacent* [5], if there at most l distinct coordinates j for which $|p_j - q_j| = 1$ and for all other coordinates j, $p_j = q_j$. A k_l-adjacency relation represents the number of points $q \in \mathbb{Z}^n$ that are adjacent to a given point $p \in \mathbb{Z}^n$. From this point of view, the k_1-adjacency on $\mathbb{Z}$ may be denoted by the number 2 and k_1-adjacent points are called *2-adjacent*. Similarly, k_1-adjacent points of $\mathbb{Z}^2$ are called *4-adjacent*; k_2-adjacent points of $\mathbb{Z}^2$ are called *8-adjacent*; and in $\mathbb{Z}^3$, k_1, k_2 and k_3-adjacent points are called *6-adjacent*, *18-adjacent* and *26-adjacent*, respectively. Let κ be an adjacency relation defined on $\mathbb{Z}^n$. *A* κ*-neighbor* of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is κ-adjacent to p. A digital image $X \subset \mathbb{Z}^n$ is κ*-connected* [15] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0,1,\ldots,r-1$. *A* κ*-component* of a digital image X is *a maximal* κ*-connected* subset of X. Let $a,b \in \mathbb{Z}$ with a<b. *A digital interval* [2] is a set of the form $[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \le z \le b\}$.

Let $(X,\kappa_0)\subset \mathbb{Z}^{n_0}$ and $(Y,\kappa_1)\subset \mathbb{Z}^{n_1}$ be digital images. A function $f:X\to Y$ is said to be (κ_0,κ_1)-*continuous* [3] if for every κ_0-connected subset U of X, f(U) is a κ_1-connected subset of Y. Such a function is called digitally continuous. A function $f:X\to Y$ is (κ_0,κ_1)-*continuous* [3] if and only if for every κ_0-adjacent points $\{x_0,x_1\}$ of X, either $f(x_0)=f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1-adjacent in Y.

By *a digital* κ*-path* [6] from x to y in a digital image X, we mean a (2,κ)-continuous function $f:[0,m]_{\mathbb{Z}}\to X$ such that f(0)=x and f(m)=y. If f(0)=f(m) then f is called *digital* κ*-loop* and the point f(0) is the base point of the loop f. A digital loop f is said to be *a trivial loop* if it is a constant function. *A simple closed* κ*-curve* of $m\geq 4$ points in a digital image X is a sequence $\{f(0),f(1),\ldots,f(m-1)\}$ of images of the κ-path $f:[0,m-1]_{\mathbb{Z}}\to X$ such that f(i) and f(j) are κ-adjacent if and only if $j=i\pm \bmod m$.

Let $(X,\kappa_0)\subset \mathbb{Z}^{n_0}$ and $(Y,\kappa_1)\subset \mathbb{Z}^{n_1}$ be digital images. A function $f:X\to Y$ is *a* (κ_0,κ_1)*-isomorphism* [1] if f is (κ_0,κ_1)-continuous and bijective and further $f^{-1}:Y\to X$ is (κ_1,κ_0)-continuous and it is denoted by $X\approx Y$.

Definition 2.1 [3]. Let $X\in \mathbb{Z}^{n_0}$ and $Y\in \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency respectively. Two (κ_0,κ_1)-continuous functions $f,g:X\to Y$ are said to be digitally (κ_0,κ_1)-homotopic in Y if there is a positive integer m and a function $H:X\times[0,m]_{\mathbb{Z}}\to Y$ such that

- for all $x\in X$, $H(x,0)=f(x)$ and $H(x,m)=g(x)$;
- for all $x\in X$, the induced function $H_x:[0,m]_{\mathbb{Z}}\to Y$ defined by

$$H_x(t)=H(x,t) \quad \text{for all } t\in[0,m]_{\mathbb{Z}},$$

is $(2,\kappa_1)$-continuous; and

- for all $t\in[0,m]_{\mathbb{Z}}$, the induced function $H_t:X\to Y$ defined by

$$H_t(x)=H(x,t) \quad \text{for all } x\in X,$$

is (κ_0,κ_1)-continuous.

The function H is called a digital (κ_0,κ_1)-homotopy between f and g. The notation $f\simeq_{\kappa_0,\kappa_1} g$ is used to indicate that functions f and g are digitally (κ_0,κ_1)-homotopic in Y. The digital (κ_0,κ_1)-homotopy relation [3] is equivalence among digitally continuous functions $f:(X,\kappa_0)\to(Y,\kappa_1)$.

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Let $f : X \rightarrow Y$ be a (κ_0, κ_1)-continuous function and let $g : Y \rightarrow X$ be a (κ_1, κ_0)-continuous function such that $f \circ g \simeq_{\kappa_1, \kappa_1}$ and $g \circ f \simeq_{\kappa_0, \kappa_0}$. Then we say X and Y have the *same* (κ_0, κ_1)*-homotopy type* [3] and that X and Y are (κ_0, κ_1)*-homotopy equivalent*.

Definition 2.2 [2].

• A digital image (X, κ) is said to be κ-contractible if its identity map is (κ, κ)-homotopic to a constant function $\overline{c}$ for some $c \in X$ where the constant function $\overline{c} : X \rightarrow X$ is defined by $\overline{c}(x) = c$ for all $x \in X$.

• We say that a (κ_0, κ_1)-continuous function $f : X \rightarrow Y$ is κ_1-nullhomotopic if f is κ_1-homotopic to a constant function $\overline{c}$ in Y.

For a digital image (X, κ) and its subset (A, κ), we call (X,A) a digital image pair with κ-adjacency. Moreover, if A is a singleton set $\{x_0\}$, then (X, x_0) is called *a pointed digital image*.

Definition 2.3 [3]. Let (X,A) be a digital image pair with κ-adjacency. Let $i : A \rightarrow X$ be the inclusion function. A is called a κ-retract of X if and only if there is a κ-continuous function $r : X \rightarrow A$ such that $r(a) = a$ for all $a \in A$. Then the function r is called a κ-retraction of X onto A.

If $f : [0, m_1]_{\mathbb{Z}} \rightarrow X$ and $g : [0, m_2]_{\mathbb{Z}} \rightarrow X$ are digital κ-paths with $f(m_1) = g(0)$, then define the product $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \rightarrow X$ [7] by

$$(f * g)(t) = \begin{cases} f(t), & 0 \leq \mathrm{t} \leq \mathrm{m}_1 \\ g(t - m_1), & \mathrm{m}_1 \leq \mathrm{t} \leq \mathrm{m}_1 + \mathrm{m}_2. \end{cases}$$

Let f and f' be κ-loops in a pointed digital image (X, x_0). We say f' is *a trivial extension* of f [3] if there are sets of κ-paths $\{f_1, f_2, \ldots, f_r\}$ and $\{F_1, F_2, \ldots, F_p\}$ in X such that

• $r \leq p$;

• $f = f_1 * f_2 * \ldots * f_r$;

• $f' = F_1 * F_2 * \ldots * F_p$;

• There are indices $1 \leq i_1 < i_2 < \ldots < i_r \leq p$ such that $F_{i_j} = f_j$, $1 \leq j \leq r$; and $i \notin \{i_1, 1_2, \ldots, i_r\}$ implies F_i is a trivial loop.

If $f, g : [0,m]_{\mathbb{Z}} \to (X,\kappa)$ are κ-paths [3] such that f(0)=g(0) and f(m)=g(m), then we say that a homotopy $H : [0,m]_{\mathbb{Z}} \times [0,M]_{\mathbb{Z}} \to X$ between f and g such that for all $t \in [0,M]_{\mathbb{Z}}$, $H(0,t) = f(0)$ and $H(m,t) = f(m)$, holds the endpoints fixed. Two loops f, f_0 with the same base point $x_0 \in X$ belong to the same loop class $[f]_X$ [3] if they have trivial extensions that can be joined by a homotopy that holds the endpoints fixed. Define $\pi_1^{\kappa}(X, x_0)$ to be the set of digital homotopy classes of κ-loops $[f]_X$ in X with base point x_0. $\pi_1^{\kappa}(X, x_0)$ is a group under the $\cdot$ product operation [3] defined by $[f]_X \cdot [g]_X = [f * g]_X$.

Lemma 2.4 [3]. Let (X, p) be a pointed digital image. Let $\overline{p} : [0,m]_{\mathbb{Z}} \to X$ be the constant function $\overline{p}(t) = p$.

Then $[\overline{p}]$ is an identity element for $\pi_1^{\kappa}(X, p)$.

Lemma 2.5 [3]. If $f : [0,m]_{\mathbb{Z}} \to X$ represents an element of $\pi_1^{\kappa}(X, p)$, then the function $g : [0,m]_{\mathbb{Z}} \to X$ defined by

$$g(t) = f(m-t) \quad \text{for } t \in [0,m]_{\mathbb{Z}}$$

is an element of $[f]^{-1}$ in $\pi_1^{\kappa}(X, p)$.

Theorem 2.6 [3]. Let X be a digital image with adjacency relation κ. If p and q belong to the same κ-component of X, then $\pi_1^{\kappa}(X, p)$ and $\pi_1^{\kappa}(X, q)$ are isomorphic groups.

A digital pointed image (X, x_0) is said to be *simply κ-connected* [4] if X is κ-connected and $\pi_1^{\kappa}(X, x_0)$ is a trivial group. If $f : (X, x_0) \to (Y, y_0)$ is a (κ_0, κ_1)-continuous map of pointed digital images, then [3]

$f_* : \pi_1^{\kappa}(X, x_0) \to \pi_1^{\kappa}(Y, y_0)$, defined by $f_*([g]) = [f \circ g]$, is a group homomorphism.

Definition 2.7 [13]. Let $c^* := (x_0, x_1, \ldots, x_n)$ be a closed κ-curve in $\mathbb{Z}^2$, where $\kappa, \overline{\kappa} = 4, 8$. A point x of the complement $\overline{c^*}$ of a closed κ-curve c^* in $\mathbb{Z}^2$ is said to be interior to c^* if it belongs to the bounded $\overline{\kappa}$-connected component of $\overline{c^*}$. The set of all interior points to c^* is denoted by $Int(c^*)$.

Since a digital image is made up of discrete points, a closure of a digital image is again itself, that is, if A is a digital image, then its closure which is denoted by $\overline{A}$ is $\overline{A} = A$.

3. Some Properties of Digital Simplicial Homology Groups

Before we give the digital version of Eilenberg-Steenrod axioms, we shall recall definitions, theorems and other important notions about [1], [8] the digital simplicial homology groups.

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Definition 3.1 [25]. Let S be a set of nonempty subset a digital image (X,κ). Then the members of S are called simplexes of (X,κ) if the following hold :

a) If p and q are distinct points of $s \in S$, then p and q are κ-adjacent.

b) If $s \in S$ and $\varnothing \neq t \subset s$, then $t \in S$.

An m-simplex is a simplex S such that $|S| = m+1$. Let P be a digital m-simplex. If P' is a nonempty proper subset of P, then P' is called a face of P. We write $Vert(P)$ to denote the vertex set of P, namely, the set of all digital 0-simplexes in P. A digital subcomplex A of a digital simplicial complex X with κ-adjacency is a digital simplicial complex [24] contained in X with $Vert(A) \subset Vert(X)$.

Let (X,κ) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some non-negative integer d. If the following statements hold then (X,κ) is called [1] a finite digital simplicial complex :

(1) If P belongs to X, then every face of P also belongs to X.

(2) If $P,Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q.

The dimension of a digital simplicial complex X is the largest integer m such that X has an m-simplex. $C_q^{\kappa}(X)$ is a free abelian group [1] with basis all digital (κ,q)-simplices in X.

Corollary 3.2 [8]. Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then for all $q > m$, $C_q^{\kappa}(X)$ is a trivial group.

Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. The homomorphism $\partial_q : C_q^{\kappa}(X) \to C_{q-1}^{\kappa}(X)$ defined (see [1]) by

$$\partial_q(< p_0, p_1, \ldots, p_q >) = \begin{cases} \sum_{i=0}^{q} (-1)^i < p_0, p_1, \ldots, \hat{p}_i, \ldots, p_q >, & \mathrm{q} \leq \mathrm{m} \\ 0, & \mathrm{q} > \mathrm{m} \end{cases}$$

is called a boundary homomorphism, where $\hat{p}_i$ means delete the point p_i.

Proposition 3.3 [1]. For all $1 \leq q \leq m$, we have $\partial_{q-1} \circ \partial_q = 0$.

Theorem 3.4 [1]. Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then

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$$C_*^{\kappa}(X):0\xrightarrow{\partial_{m+1}}C_m^{\kappa}(X)\xrightarrow{\partial_m}C_{m-1}^{\kappa}(X)\xrightarrow{\partial_{m-1}}\ldots\xrightarrow{\partial_1}C_0^{\kappa}(X)\xrightarrow{\partial_0}0$$

is a chain complex.

Definition 3.5 [15]. Let (X,κ) be a digital simplicial complex.

(1) $Z_q^{\kappa}(X)=\text{Ker }\partial_q$ is called the group of digital simplicial q-cycles.

(2) $B_q^{\kappa}(X)=\text{Im }\partial_{q+1}$ is called the group of digital simplicial q-boundaries.

(3) $H_q^{\kappa}(X)=Z_q^{\kappa}(X)/B_q^{\kappa}(X)$ is called the qth digital simplicial homology group.

Because we need to use relative simplicial homology for digital images, we give its definition. Let (A,κ) be a digital subcomplex of the digital simplicial complex (X,κ). The chain group $C_q^{\kappa}(A)$ is a subgroup of the chain group $C_q^{\kappa}(X)$. The quotient group $C_q^{\kappa}(X)/C_q^{\kappa}(A)$ is called the group of relative chains of X modulo A and is denoted by $C_q^{\kappa}(X,A)$. The boundary operator $\partial_q : C_q^{\kappa}(A)\to C_{q-1}^{\kappa}(A)$ is the restriction of the boundary operator on $C_q^{\kappa}(X)$. A homomorphism $C_q^{\kappa}(X,A)\to C_{q-1}^{\kappa}(X,A)$ of the relative chain groups is induced by ∂_q and is also denoted by ∂_q.

Definition 3.6 [21]. Let (A,κ) be a digital subcomplex of the digital simplicial complex (X,κ).

- $Z_q^{\kappa}(X,A)=\text{Ker }\partial_q$ is called the group of digital relative simplicial q-cycles.
- $B_q^{\kappa}(X,A)=\text{Im }\partial_{q+1}$ is called the group of digital relative simplicial q-boundaries.
- $H_q^{\kappa}(X,A)=Z_q^{\kappa}(X)/B_q^{\kappa}(X)$ is called the qth digital relative simplicial homology group.

Theorem 3.7 [8]. Let (X,κ) be a directed digital simplicial complex of dimension m.

(1) $H_q^{\kappa}(X)$ is a finitely generated abelian group for every $q\geq 0$.

(2) $H_q^{\kappa}(X)$ is a trivial group for all $q>m$.

(3) $H_m^{\kappa}(X)$ is a free abelian group, possible zero.

Theorem 3.8 [8]. For each $q \geq 0$, H_q^{κ} is a covariant functor from the category of digital simplicial complexes and simplicial maps to the category of abelian groups.

Corollary 3.9 [1]. If $f:(X,\kappa_1) \to (Y,\kappa_2)$ is a digitally (κ_1,κ_2)-isomorphism, then $f_*: H_q^{\kappa_1}(X) \to H_q^{\kappa_2}(Y)$ is a group isomorphism.

Let $\varphi:(X,\kappa_0) \to (Y,\kappa_1)$ be a function between digital images. If for every digital (κ_0,m)-simplex P determined by the adjacency relation κ_0 in X, $\varphi(P)$ is a (κ_1,n)-simplex in Y for some $n \leq m$, then φ is called (see [8]) a digital simplicial map. Let $\varphi:(X,\kappa_0) \to (Y,\kappa_1)$ be a digital simplicial map. For $q \geq 0$, we define a homomorphism $\varphi_é: C_q^{\kappa_0}(X) \to C_q^{\kappa_1}(Y)$ by

$$\varphi_é(< p_0,\ldots,p_q >) =< \varphi_é(p_0),\ldots,\varphi_é(p_q) > .$$

Theorem 3.10 [26]. Let $f,g:(X,\kappa_1) \to (Y,\kappa_2)$ be the digitally (κ_1,κ_2)-continuous functions. Assume that there are homomorphisms $\psi_q: C_q^{\kappa_1}(X) \to C_{q+1}^{\kappa_2}(Y)$ such that $f_é - g_é = \partial'_{q+1} \circ \psi_q + \psi_{q-1} \circ \partial_q$ where $\partial_q: C_q^{\kappa_1}(X) \to C_{q-1}^{\kappa_1}(X)$ and $\partial'_{q+1}: C_{q+1}^{\kappa_2}(Y) \to C_q^{\kappa_2}(Y)$ are digital boundary operators of (X,κ_1) and (Y,κ_2), respectively. Then $f_* = g_*: H_q^{\kappa_1}(X) \to H_q^{\kappa_2}(Y)$ as group homomorphisms.

Theorem 3.11 [26]. Let (A,κ) be a nonempty subset of digital image (X,κ) with κ-adjacency, and let

$i:(A,\kappa) \to (X,\kappa)$ be the inclusion. Then $i_*: H_q^{\kappa}(A) \to H_q^{\kappa}(X)$ is a monomorphism for every $q \geq 0$.

4. The Eilenberg-Steenrod Axioms

We now list the digital version of Eilenberg-Steenrod axioms which is very significant in algebraic topology. Note that the first 5 axioms hold in digital images. Since these axioms are proved in a similar way in algebraic topology, we don't prove these axioms.

Axiom 1 (Identity axiom) [21]. Let X be a digital image with κ-adjacency. If $i:(X,\kappa) \to (X,\kappa)$ is the identity, then $i_*: H_*^{\kappa}(X) \to H_*^{\kappa}(X)$ is the identity.

Axiom 2 (Composition axiom) [21]. Let X, Y and Z be digital images with κ_0, κ_1 and κ_2-adjacency, respectively. If $h:(X,\kappa_0) \to (Y,\kappa_1)$ and $k:(Y,\kappa_1) \to (Z,\kappa_2)$ are digitally continuous functions, then $(k \circ h)_* = k_* \circ h_*$.

Axiom 3 (Commutativity axiom) [21]. Let (X,A) and (Y,B) be digital image pairs with κ_0 and κ_1-adjacency, respectively. If $f:(X,A) \to (Y,B)$, then the following diagram commutes :

$$\begin{tikzcd} H_q^{\kappa_0}(X,A) \arrow[r, "f_*"] \arrow[d, "\partial_*"'] & H_q^{\kappa_1}(Y,B) \arrow[d, "\partial_*"] \\ H_{q-1}^{\kappa_0}(A) \arrow[r, "(f|_A)_*"] & H_{q-1}^{\kappa_1}(B) \end{tikzcd}$$

Axiom 4 (Exactness axiom) [21]. The sequence

$$\ldots \to H_q^{\kappa}(A) \xrightarrow{i_*} H_q^{\kappa}(X) \xrightarrow{p_*} H_q^{\kappa}(X,A) \xrightarrow{\partial_*} H_{q-1}^{\kappa}(A) \to \ldots$$

is exact, where $i : A \to X$ and $p : X \to (X,A)$ are inclusion maps.

Axiom 5 (Dimension axiom) [1]. If X is a one-point space with κ-adjacency, then $H_q^{\kappa}(X) = 0$ for $q \neq 0$ and $H_0^{\kappa}(X) \cong \mathbb{Z}$.

Homotopy and excision axioms don't hold in digital images. We first give definition of these axioms in algebraic topology.

Homotopy axiom [21]. Let X and Y be digital images with κ_1 and κ_2-adjacency, respectively. If

$h : (X,\kappa_1) \to (Y,\kappa_2)$ and $k : (X,\kappa_1) \to (Y,\kappa_2)$ are homotopic, then $h_* = k_*$.

Excision axiom [21]. Given (X,A), let U be an open subset of X such that $\bar{U} \subset IntA$, then inclusion induces an isomorphism $H_q^{\kappa}(X-U, A-U) \cong H_q^{\kappa}(X,A)$.

Proposition 4.1 Excision axiom for simplicial homology doesn't be hold in digital images.

Proof. Let $X = \{c_0 = (0,0), c_1 = (1,0), c_2 = (1,1), c_3 = (0,1)\} \subset \mathbb{Z}^2$ be a digital image with 4-adjacency (see Figure 1).

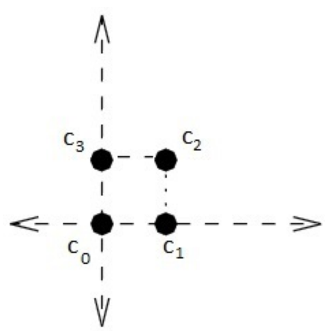

Figure 1. X

Since X is a digital simple closed 4-curve (see [1]), the digital simplicial homology groups of X are

$$H_q^4(X) = \begin{cases} \mathbb{Z}, & \text{q=0,1} \\ 0, & q \neq 0,1. \end{cases}$$

Let $A = \{c_0 = (0,0), c_1 = (1,0), c_2 = (1,1)\}$ and $U = \{c_0 = (0,0)\}$ (see Figure 2).

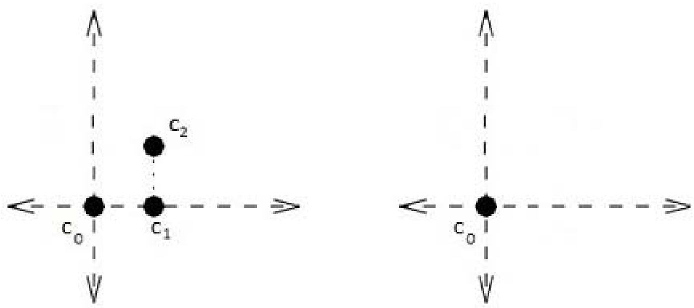

Figure 2. A and U

It's clear that $A \subset X$, $U \subset A$ and $\bar{U} \subset IntA$. The digital simplicial homology groups of A and U are

$$H_q^4(A) = H_q^4(U) = \begin{cases} \mathbb{Z}, & \text{q=0} \\ 0, & q \neq 0. \end{cases}$$

So we get following exact sequence :

$$\ldots \to H_q^4(A) \to H_q^4(X) \to H_q^4(X,A) \to H_{q-1}^4(A) \to \ldots$$

For q>1, as $H_q^4(A) = 0$ we get $H_q^4(X) \cong H_q^4(X,A)$. For q=0,1 we have following exact sequence :

$$0 \xrightarrow{i} \mathbb{Z} \xrightarrow{j} H_1^4(X,A) \xrightarrow{k} \mathbb{Z} \xrightarrow{l} \mathbb{Z} \xrightarrow{m} H_0^4(X,A) \xrightarrow{n} 0$$

From this exact sequence, $\operatorname{Im} m = \operatorname{Ker} n = H_0^4(X,A)$, thus m is an epimorphism and $H_0^4(X,A) \cong \mathbb{Z}$. As l is an inclusion map, from Theorem 3.11 we get that $\operatorname{Ker} l = 0$. $\operatorname{Im} k = 0$ due to exact sequence. From first isomorphism theorem, $H_1^4(X,A) / \operatorname{Ker} k \cong \operatorname{Im} k$ and we find that $H_1^4(X,A) = \operatorname{Ker} k$ since $\operatorname{Im} k = 0$. As this sequence is exact, we have $\operatorname{Ker} k = \operatorname{Im} j$ and so we get $\operatorname{Im} j = H_1^4(X,A)$. Thus j is an epimorphism. On the other hand, from exactness of sequence, we have $\operatorname{Im} i = \operatorname{Ker} j$. Since $\operatorname{Im} i = 0$, $\operatorname{Ker} j = 0$ and thus j is a monomorphism. Therefore j is an isomorphism and we find that $H_1^4(X,A) \cong \mathbb{Z}$. Consequently, we obtain

$$H_q^4(X,A)=\begin{cases}\mathbb{Z}, & \text{q=0,1}\\ 0, & \text{q} \neq 0,1.\end{cases}$$

Now we compute $H_q^4(X-U,A-U)$. $X-U=\{c_1=(1,0),c_2=(1,1),c_3=(0,1)\}$ and $A-U=\{c_1=(1,0),c_2=(1,1)\}$ (see Figure 3).

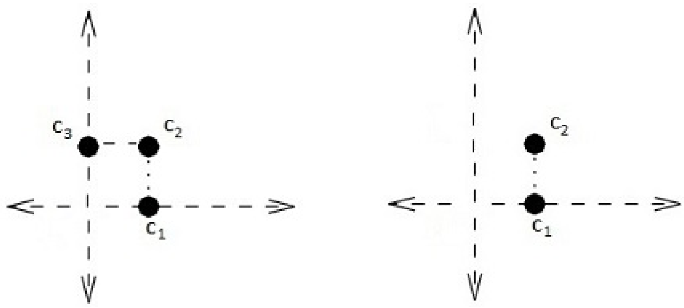

Figure 3. X-U and A-U

It's clear that $A-U\subset X-U$. The digital simplicial homology groups of X-U and A-U are

$$H_q^4(X-U)=H_q^4(A-U)=\begin{cases}\mathbb{Z}, & \text{q=0}\\ 0, & \text{q} \neq 0.\end{cases}$$

So we get following exact sequence :

$$\ldots\to H_q^4(A-U)\to H_q^4(X-U)\to H_q^4(X-U,A-U)\to H_{q-1}^4(A-U)\to\ldots$$

For q>1, as $H_q^4(A-U)=0$ we get $H_q^4(X-U)\cong H_q^4(X-U,A-U)$. For q=0,1 we have following exact sequence :

$$0\xrightarrow{\alpha} H_1^4(X-U,A-U)\xrightarrow{\beta}\mathbb{Z}\xrightarrow{\gamma}\mathbb{Z}\xrightarrow{\delta} H_0^4(X-U,A-U)\xrightarrow{\lambda}0$$

From this exact sequence, $\operatorname{Im}\delta=\operatorname{Ker}\lambda=H_0^4(X-U,A-U)$ thus δ is an epimorphism and $H_0^4(X-U,A-U)=\mathbb{Z}$. As γ is an inclusion map, from Theorem 3.11 we get that $\operatorname{Ker}\gamma=0$. $\operatorname{Im}\beta=0$ because of exact sequence. From first isomorphism theorem,

$$H_1^4(X-U,A-U)/\operatorname{Ker}\beta\cong\operatorname{Im}\beta$$

and we find that $H_1^4(X-U,A-U)=\operatorname{Ker}\beta$ since $\operatorname{Im}\beta=0$. As this sequence is exact, we have

$\text{Ker}\ \beta = \text{Im}\ \alpha = 0$ and so we get $H_1^4(X, A) = 0$. As a result, we obtain

$$H_q^4(X-U, A-U) = \begin{cases} \mathbb{Z}, & \text{q=0} \\ 0, & \text{q} \neq 0. \end{cases}$$

For q=1, since $H_q^4(X-U, A-U)$ ® $H_q^4(X, A)$, excision theorem for simplicial homology doesn't be hold in digital images. □

We can also say that the homotopy axiom for simplicial homology doesn't hold in digital images. For this purpose, we must give a counterexample to homotopy axiom for digital images. The simple closed curve X of Proposition 4.1 is 4-contractible (see [1]) so we have $H_1^4(X) \cong H_1^4(\{x_0\}) \cong 0$. But from Proposition 4.1 we would have $H_1^4(X) \cong \mathbb{Z}$. Therefore $X \simeq_{(4,4)} \{x_0\}$ that is X and singleton set $\{x_0\}$ are digitally $(4,4)$ - homotopy equivalent, but they don't have isomorphic homology groups.

5. Universal Coefficient Theorem for Digital Simplicial Homology

We state universal coefficient theorem for homology of a digital simplicial complex.

Theorem 5.1 [21]. Let (X, κ) be a digital simplicial complex. For any abelian group G, there is exact sequence

$$0 \to H_q^\kappa(X) \otimes G \to H_q^\kappa(X, G) \to Tor(H_{q-1}^\kappa(X), G) \to 0,$$

where $H_q^\kappa(X) = H_q^\kappa(X; \mathbb{Z})$. This exact sequence is split; hence

$$H_q^\kappa(X, G) \cong H_q^\kappa(X) \otimes G \oplus Tor(H_{q-1}^\kappa(X), G).$$

Example 5.2 Let $MSC_8^{'} = \{c_0 = (-1,0), c_1 = (0,-1), c_2 = (0,1), c_3 = (1,0)\} \subset \mathbb{Z}^2$ be a digital image with 8-adjacency (see Figure 4).

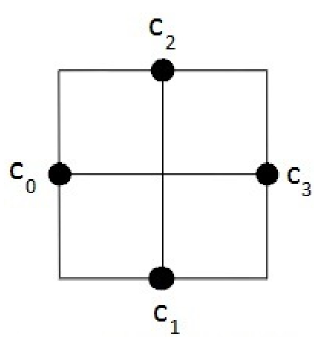

Figure 4. [14] $MSC_8^{'}$

The digital simplicial homology groups of $MSC_8^{'}$ are

$$H_q^8(MSC_8^{'}) = \begin{cases} \mathbb{Z}, & \text{q=0,1} \\ 0, & \text{q} \neq 0,1 \end{cases}$$

(see [8]). We use universal coefficient theorem for homology to calculate simplicial homology groups of $MSC_8^{'}$ with $\mathbb{Z}_3$-coefficient. For q=0, we have $H_0^8(MSC_8^{'};\mathbb{Z}_3) \cong \mathbb{Z}_3$. For q=1, we have $H_1^8(MSC_8^{'};\mathbb{Z}_3) \cong \mathbb{Z}_3$. For all $q \geq 2$, we get $H_q^8(MSC_8^{'};\mathbb{Z}_3) \cong 0$. As a result we find that

$$H_q^8(MSC_8^{'};\mathbb{Z}_3) = \begin{cases} \mathbb{Z}_3, & \text{q=0,1} \\ 0, & \text{q} \neq 0,1. \end{cases}$$

6. Künneth Formula for Digital Simplicial Homology

We recall that Künneth formula for simplicial homology used in algebraic topology.

Theorem 6.1 [21]. Suppose X and Y are topological spaces. We then have the following relation between the homology groups of X, Y and the product space $X \times Y$. For any $n \geq 0$ and any module M over a principal ideal domain R, we have:

$$H_n(X \times Y; M) \cong \Big(\sum_{i+j=n} H_i(X;M) \otimes H_j(Y;M) \Big) \oplus \Big(\sum_{p+q=n-1} Tor(H_p(X;M), H_q(Y;M)) \Big)$$

where Tor denotes the Tor functor. If $M = \mathrm{F}$ is a field, then the Tor functor is always trivial and in this case Künneth formula can be stated as

$$H_n(X \times Y; \mathrm{F}) \cong \sum_{i+j=n} H_i(X;\mathrm{F}) \otimes H_j(Y;\mathrm{F})$$

for any $n \geq 0$.

We show that Künneth formula for simplicial homology doesn't be hold in digital images.

Proposition 6.2 Künneth formula for simplicial homology doesn't be hold in digital images.

Proof. Let $X = [0,1]_{\mathbb{Z}}$ and $Y = [0,1]_{\mathbb{Z}} \times [0,1]_{\mathbb{Z}}$ be digital images with 2-adjacency and 4-adjacency, respectively. Assume that $M = \mathbb{Z}$. Then simplicial homology groups of X are

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$$H_n^2(X) = \begin{cases} \mathbb{Z}, & \text{n=0} \\ 0, & \text{n} \neq 0. \end{cases}$$

Since Y is a digital simple closed 4-curve (see [1]), its digital simplicial homology groups are

$$H_n^4(Y) = \begin{cases} \mathbb{Z}, & \text{n=0,1} \\ 0, & \text{n} \neq 0,1. \end{cases}$$

Since $X \times Y$ is $MSS_6^{'}$, its simplicial homology groups are

$$H_n^6(X \times Y) = \begin{cases} \mathbb{Z}, & \text{n=0} \\ \mathbb{Z}^5, & \text{n=1} \\ 0, & \text{n} \neq 0,1 \end{cases}$$

(see [8]). We now use Künneth formula to compute $H_n^6(X \times Y)$ for $n \geq 0$. Since $M = \mathbb{Z}$ is a field, $H_n^6(X \times Y; \mathbb{Z}) \cong \sum_{i+j=n} H_i^2(X; \mathbb{Z}) \otimes H_j^4(Y; \mathbb{Z})$. For n=0, we have

$$H_0^6(X \times Y; \mathbb{Z}) \cong H_0^2(X; \mathbb{Z}) \otimes H_0^4(Y; \mathbb{Z}) \cong \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}.$$

For n=1, we have

$$H_1^6(X \times Y; \mathbb{Z}) \cong (H_0^2(X; \mathbb{Z}) \otimes H_1^4(Y; \mathbb{Z})) \oplus (H_1^2(X; \mathbb{Z}) \otimes H_0^4(Y; \mathbb{Z})) \cong (\mathbb{Z} \otimes \mathbb{Z}) \oplus (0 \otimes \mathbb{Z}) \cong \mathbb{Z}.$$

For all $n \geq 2$, we have $H_n^6(X \times Y; \mathbb{Z}) \cong 0$. Therefore we find that

$$H_n^6(X \times Y; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{n=0,1} \\ 0, & \text{n} \neq 0,1. \end{cases}$$

However homology group of $X \times Y$ for n=1 must be $H_1^6(X \times Y; \mathbb{Z}) = \mathbb{Z}^5$, so we get a contradiction. As a result, Künneth formula for simplicial homology doesn't be hold in digital images. $\square$

7. The Hurewicz Theorem Need Not Be Hold In Digital Images

Theorem 7.1 (Hurewicz Theorem) [24]. If X is path connected connected, then the Hurewicz map $\varphi : \pi_1(X, x_0) \to H_1(X)$ is a surjection with $\pi_1(X, x_0)^{'}$, the commutator subgroup of $\pi_1(X, x_0)$. Hence

$$\pi_1(X, x_0) / \pi_1(X, x_0)^{'} \cong H_1(X).$$

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We now give an example which shows that the Hurewicz Theorem theorem needn't be hold in digital images.

Example 7.2 Let

$$MSS_{18} = \{c_0 = (0,0,1), c_1 = (1,1,1), c_2 = (1,2,1), c_3 = (0,3,1), c_4 = (-1,2,1), c_5 = (-1,1,1), c_6 = (0,1,0),$$

$$c_7 = (0,2,0), c_8 = (0,2,2), c_9 = (0,1,2)\} \subset \mathbb{Z}^3$$

be a digital image with 18-adjacency (see Figure 5).

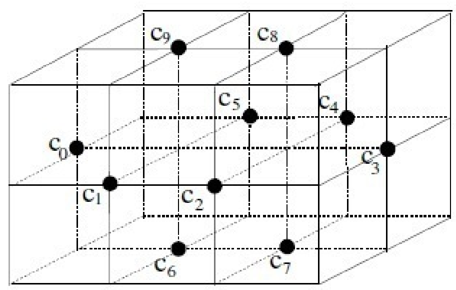

Figure 5. [14] MSS_{18}

Since MSS_{18} is 18-contractible minimal simple closed 18-surface, $\pi_1^{18}(MSS_{18}, c_0) \cong \{0\}$. However

$$H_q^{18}(MSS_{18}) = \begin{cases} \mathbb{Z}, & \text{q=0} \\ \mathbb{Z}^3, & \text{q=1} \\ 0, & \text{q} \neq 0,1 \end{cases}$$

(see [8]). $0 = \pi_1^{18}(MSS_{18}, c_0) / \pi_1^{18}(MSS_{18}, c_0)' \rightarrow H_1^{18}(MSS_{18}) = \mathbb{Z}^3$.

So we can state the following.

Proposition 7.3 Hurewicz Theorem does not be hold in digital images.

8. Conclusion

The aim of this article is to determine differences between digital topology and algebraic topology. We state Eilenberg-Steenrod axioms for the simplicial homology groups of digital images. Especially we present that excision axiom which is the one of Eilenberg-Steenrod axioms that doesn't exist in digital images. Universal Coefficient theorem and Künneth formula for digital simplicial homology are presented with examples. Also we show that the Hurewicz Theorem need not be hold in digital images. Digital simplicial homology is needed to study digital cohomology groups and digital cohomology operations which are very important in algebraic topology and stable homotopy theory. We hope that this theory will help to the analysis and understanding of these topics.

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